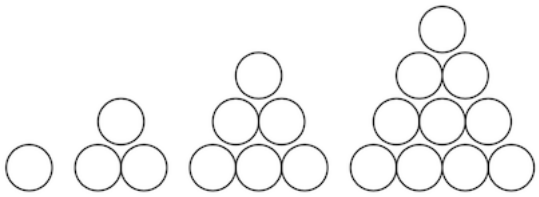
Grade 9/10 Math Circles

February 15

The Shape of You - Problem Set

Please put on your Real 3D™ glasses

1. Here's one possible way to keep track of the numbers in the 3D Pascal's Triangle:



(Write down the number of ways to get from the top to a particular intersection in each layer, always moving downwards, in the corresponding circles).

2. Here's something interesting I just noticed though: Is there a pattern to the *number of* circles in each layer above? Can you predict how many circles there will be in the next couple of layers, even without drawing all the circles?

3. Where have we seen the above sequence of numbers $(1, 3, 6, 10, \ldots)$ before? Can you find (and prove!) a formula for the above sequence of numbers (that goes $1, 3, 6, 10, \ldots$?)

4. Say we decided to build a four dimensional Pascal's Triangle. How would you need to keep track of the numbers in a 4D Pascal's Triangle? Can you try to predict what those numbers would be using polynomials?

5. If we decided to count the number of circles in "layers" of tetrahedrons[1], what sequence of numbers would we get? What do you think the pattern would be?

[1] Hmm... this might be a hint for the previous question... aren't I clever for planning out the questions like this?

Formula-One

6. Last week, we might have seen the *closed form* formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

(If this looks unfamiliar, or you don't know how to prove this, check out the Week 1 Problem Set!). Using the same "Word Problem" interpretation as in Week 1, try to find a (similar) formula for the numbers in our 3D Pascal's Triangle!

Note: I didn't give you a symbol to use as notation for these numbers[2], like I gave you the symbol $\binom{n}{k}$, so as a good first step you could come up with a symbol on your own! Remember, notation is just a *mathematician's shorthand*, so there's no right answer. You only need to find a shorthand that makes sense for you.

Bonus: Does your symbol generalize nicely to higher dimensions?

[2]There's a very good reason for this: there's no widespread symbol for trinomial expansion coefficients.